A note on rational surfaces in projective four-space.

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1. Introduction

A few years ago Ellingsrud and Peskine proved ([12]) that there exist only finitely many irreducible components of the Hilbert scheme of $\mathbb{P}^4$ parametrizing smooth surfaces not of general type; in particular, as conjectured by Hartshorne and Lichtenbaum, the degree of smooth rational surfaces $S \subset \mathbb{P}^4$ is bounded. This result has been successively improved ([5], [8], [4], [9]) and today it is believed that if $S \subset \mathbb{P}^4$ is of non general type, then $deg(S) \leq 15$; also no rational surface of degree $d > 12$ is known.

In this note we consider rational surfaces $S \subset \mathbb{P}^4$ ruled by cubics and quartics (i.e. possessing a base point free pencil of cubic or quartic rational curves) and we prove that such a surface has $deg(S) \leq 12$. (We recall that the classification of scrolls and conic bundles is known [3], [11], [6], [1]).

The proof uses ad-hoc arguments which (unfortunately) do not seem to generalize.

Using this result we then prove that if $S \subset \mathbb{P}^4$ is the image of a blow-up of $\mathbb{F}_n$ embedded by a linear system of the form $aC_0 + bf - E_1 - ... - E_r$ (in the sequel, we will call such a linear system a "linear system on $\mathbb{F}_n$ with simple base points") then, again, $deg(S) \leq 12$.

2. Generalities

Let $S \subset \mathbb{P}^4$ be a smooth, non-degenerated, rational surface. If S is isomorphic to $\mathbb{P}^2$ then, by Severi's theorem, S is a Veronese surface. If $S \simeq \mathbb{F}_n$ then S is geometrically ruled and it is not difficult to see that $n = 1$ and S is a cubic scroll. Hence we may assume that S is isomorphic to a blow-up of some $\mathbb{F}_n, n \geq 0$.

Definition 1. *We will say that S is a-ruled if there exists on S a base point free pencil of rational curves of degree a in $\mathbb{P}^4$.*

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Remark 1. *Such a pencil yields a morphism $p : S \to \mathbb{P}^1$ which presents S as ruled by the curves of the pencil. Of course the same S might be a-ruled for different values of a.*

Notice that since S is not geometrically ruled, there is at least one singular fiber.

Lemma 1. *Let $S \subset \mathbb{P}^4$ be a smooth, rational a-ruled surface, $a \geq 3$. If the general fiber of $p : S \to \mathbb{P}^1$ is degenerated in $\mathbb{P}^4$, then S contains a plane curve of degree $d - a$, residual to a fiber in an hyperplane section.*

Proof: Let x be a general point of $\mathbb{P}^1$. The fiber f_x is a smooth rational curve of degree a in $\mathbb{P}^4$. By assumption f_x is contained in an hyperplane, H_x (note that H_x is uniquely determined because f_x is not a plane curve since $a \geq 3$). Let C_x denote the residual curve: $C_x \sim H_x - f_x$. Since two general fibers are linearly equivalent, we have $C_x \sim C_y$ (they are both sections of $\mathcal{O}_S(1) \otimes p^*\mathcal{O}_{\mathbf{P}^1}(-1)$). Since S is linearly normal (Severi's theorem) and since f_x is not a plane curve, $h^0(\mathcal{O}_S(1 - f_x)) = 1$. It follows that $C_x = C_y$. Now $C_x \subset H_x \cap H_y$, and since S is non-degenerated, we may assume $H_x \neq H_y$, hence C_x is a plane curve of degree $d - a$. ∎

The next proposition will be used several time in the sequel:

Proposition 2. *Let $S \subset \mathbb{P}^4$ be a smooth, non-degenerated, surface of degree d, not of general type. If $d \geq 9$, then $h^0(\mathcal{I}_S(3)) = 0$; in particular if $d > 9$ then $\pi \leq G(d, 4)$ where π is the sectional genus of S and where $G(d, 4)$ denotes the maximal genus of smooth degree d curves in $\mathbb{P}^3$ not lying on a cubic surface.*

Proof: See [10] ∎

Remark 2. *If $d > 12$, then $G(d, 4) = 1 + \frac{d^2 - 3r(4-r)}{8}$ where $d + r \equiv 0 (mod 4)$ and $0 \leq r < 4$. In particular $\pi \leq 1 + \frac{d^2}{8}$; moreover if equality occurs then $\pi = G(d, 4)$ and the general hyperplane section of S is a.C.M. (arithmetically Cohen-Macaulay), but this is impossible because an a.C.M. surface in $\mathbb{P}^4$ not of general type has $d \leq 8$ (see [10]).*

In conclusion if $d > 12$ and S is not of general type then $\pi < 1 + \frac{d^2}{8}$.

Corollary 3. *Let $S \subset \mathbb{P}^4$ be a smooth, a-ruled, rational surface. Assume $a \geq 3$. If the general fiber of $p : S \to \mathbb{P}^1$ is degenerated, then:*
(i) $\pi = \frac{(d-a-1)(d-a-2)}{2} + a - 1$.
(ii) $1 + 2a - \sqrt{2a^2 - 6a + 5} \leq d \leq 1 + 2a + \sqrt{2a^2 - 6a + 5}$.
(iii) if $d > 12$, *then* $\frac{4a + 6 - 2\sqrt{a^2 - 3a + 15}}{3} < d < \frac{4a + 6 + 2\sqrt{a^2 - 3a + 15}}{3}$.

Proof: (i) From lemma 1 it follows that $H \sim C+f$ where C is a plane curve of degree $d-a$ and where f is a rational curve of degree a. Since $a = f.H = f.C$, we get: $\pi = p_a(C \cup f) = p_a(C) + p_a(f) + a - 1 = \frac{(d-a-1)(d-a-2)}{2} + a - 1$.

(ii) The general hyperplane section of S is non-degenerated in $\mathbb{P}^3$ so its genus has to satisfy Castelnuovo's inequality: $\pi \leq (\frac{d}{2} - 1)^2$. Combining with (i) yields: $d^2 + 2d(-1-2a) + 2a^2 + 10a - 4 \leq 0$, and the result follows.

(iii) By Remark 2: $\pi < 1 + \frac{d^2}{8}$, combining with (i) gives: $3d^2 + 2d(-4a - 6) + 4a^2 + 20a - 8 < 0$, and we conclude. ∎

3. a-ruled rational surfaces with $a \leq 3$.

For sake of completeness we recall the following:

Proposition 4. *Let $S \subset \mathbb{P}^4$ be a smooth, non degenerated, rational surface.*

(i) if S is a scroll ($a = 1$), then S is a cubic scroll.

(ii) if S is ruled in conics ($a = 2$), then either S is a Del Pezzo surface ($d = 4$), or S is a Castelnuovo surface ($d = 5$).

Proof: For (i) see [3], for (ii) see [11], [6] ∎

Proposition 5. *Let $S \subset \mathbb{P}^4$ be a smooth rational surface ruled in cubics ($a = 3$).*

(i) $5 \leq d \leq 9$

(ii) the possibilities for (d, π) are: $(5, 2), (6, 3), (7, 5), (8, 8), (9, 12)$.

Proof: Since the fibers are cubics we can apply Corollary 3. From (ii) we get $5 \leq d \leq 9$, then we compute π with (i). ∎

4. Rational surfaces ruled in quartics.

Lemma 6. *Let $S \subset \mathbb{P}^4$ be a smooth rational surface ruled in quartics. If the general fiber of $p : S \to \mathbb{P}^1$ is non-degenerated, then $h^1(\mathcal{O}_S(1)) = 0$ and $d \leq 9$.*

Proof: Consider Euler's sequence:

$$0 \to M_S \to V \otimes \mathcal{O}_S \xrightarrow{\rho} \mathcal{O}_S(1) \to 0$$

($M := \Omega_{\mathbf{P}^4}(1)$).

We want to apply p_* to this exact sequence. Restricting to a fiber we have:

$$0 \to M_{f_x} \to V \otimes \mathcal{O}_{f_x} \xrightarrow{\rho_x} \mathcal{O}_{f_x}(1) \to 0$$

Notice that $h^0(\mathcal{O}_{f_x}(1)) = 5$ and $h^1(\mathcal{O}_{f_x}(1)) = 0$ for every x in $\mathbb{P}^1$ (even if f_x is singular); by base change it follows that $p_*(\mathcal{O}_S(1))$ is a rank 5 vector bundle on $\mathbb{P}^1$ and $R^ip_*(\mathcal{O}_S(1)) = 0, i > 0$. Moreover, since for general x, f_x spans $\mathbb{P}^4$, ρ_x is an isomorphism and $h^0(M_{f_x}) = 0$ for general x. This implies $p_*(M_S) = 0$ (it would be a torsion subsheaf of $p_*(V \otimes \mathcal{O}_S) = 5.\mathcal{O}_{\mathbf{P}^1}$). Hence we get an injection: $0 \to 5.\mathcal{O}_{\mathbf{P}^1} \to p_*(\mathcal{O}_S(1))$; let T denote the cokernel, T has finite support (it has rank zero). Taking cohomology in the exact sequence:

$$0 \to 5.\mathcal{O}_{\mathbf{P}^1} \to p_*(\mathcal{O}_S(1)) \to T \to 0$$

and since $h^0(p_*(\mathcal{O}_S(1)) = h^0(\mathcal{O}_S(1)) = 5$ by Severi's theorem, we have $h^0(T) = 0$, hence $T = 0$ and $5.\mathcal{O}_{\mathbf{P}^1} \simeq p_*(\mathcal{O}_S(1))$. It follows that $h^1(p_*(\mathcal{O}_S(1)) = 0$. Since $R^ip_*(\mathcal{O}_S(1)) = 0, i > 0$, by Leray's spectral sequence $h^1(\mathcal{O}_S(1)) = h^1(p_*(\mathcal{O}_S(1)) = 0$ and S is non-special.

As shown in [2], non-special rational surfaces have $d \leq 9$. ∎

Remark 3. *Non-special rational surfaces are classified in [2].*

Proposition 7. *Let $S \subset \mathbb{P}^4$ be a smooth rational surface ruled in quartics, then $d \leq 12$.*

Proof: If the general fiber f_x is a non-degenerated quartic in $\mathbb{P}^4$, we conclude with the previous proposition. If f_x is degenerated, we conclude with Corollary 3. ∎

Remark 4. *As claimed in [7], every known rational surface contains a plane curve.*

Linear systems with simple base points on F_n.

In this section we consider rational surfaces which are images of $\mathbb{F}_n$ by linear systems with simple base-points.

Notations: Let $S \subset \mathbb{P}^4$ be a smooth, non degenerated, surface isomorphic to $\mathbb{F}_n$ blown-up at r points $y_1, ..., y_r$.

We have $Pic(\mathbb{F}_n) = C_0'\mathbf{Z} \oplus f'\mathbf{Z}$ where $(C_0')^2 = -n$. Denoting by C_0, f the strict transform of C_0', f', we have $Pic(S) = C_0\mathbf{Z} \oplus f\mathbf{Z} \oplus E_1\mathbf{Z} \oplus ... \oplus E_r\mathbf{Z}$. We will work under the following assumptions:

$$(*) \begin{cases} (a) & \text{the } y_i\text{'s lie in different fibers of } \pi : \mathbb{F}_n \to \mathbb{P}^1 \\ (b) & \text{If } n \geq 1, \text{ no } y_i \text{ lies on } C_0' \\ (c) & H \sim aC_0 + bf - E_1 - ... - E_r \text{ (”simple base points on } \mathbb{F}_n\text{”)} \end{cases}$$

Remark 5. *It follows that S is a-ruled and that the fibers of the ruling $S \to \mathbb{P}^1$ have at most two irreducible components.*

The intersection theory on S is given by: $C_0^2 = -n, C_0E_i = 0, C_0f = 1, f^2 = 0, fE_i = 0, E_iE_j = \delta_{ij}$.
The canonical class is $K_S \sim -2C_0 - (n+2)f + \Sigma E_i$.

We have the relations:
1) $H^2 = d$
2) $2\pi - 2 = H(H+K)$
3) $d(d-5) - 10(\pi - 1) + 12\chi = 2K^2$

After some computations we get:
1) $d = -a^2n + 2ab - r$
2) $2\pi - 2 = -a^2n + an - 2a + 2ab - 2b$
3) $d(d-5) - 10(\pi - 1) = 4 - 2r$

Lemma 8. *With notations as above, if $\pi < \frac{d^2}{8}$, then $a \leq 9$.*

Proof: From 1): $r = -a^2n + 2ab - d$, inserting in 3): $d^2 - 7d + 3a^2n - 5an + 10a - 4 + b(10 - 6a) = 0$, i.e.

$$b = \frac{d^2 - 7d + 3a^2n - 5an + 10a - 4}{6a - 10} \qquad (*)$$

Using 2): $\pi - 1 = -\frac{an}{2}(a-1) - a + \frac{(a-1)(d^2-7d+3a^2n-5an+10a-4)}{6a-10}$

Now, using this expression of $\pi - 1$ in the inequality $\pi - 1 < \frac{d^2}{8}$, yields $f_a(d) < 0$ (**), where:

$$f_a(d) = d^2(a+1) - 28(a-1)d + 16a^2 - 16a + 16$$

Notice that n has disappeared!

We have $\frac{\partial f_a(d)}{\partial d} = 0 \Leftrightarrow d = \frac{14(a-1)}{a+1} =: d_0$. Now $f_a(d_0) = (a-1)(16a - \frac{196(a-1)}{a+1}) + 16$. If $a \geq 10$, we have $f_a(d) \geq f_a(d_0) > 0, \forall d$, contradicting (**). (indeed $(16a - \frac{196(a-1)}{a+1} > 0$ if $a \geq 11$ and one checks directly that $f_{10}(d_0) > 0$.)

In conclusion, if $\pi < 1 + \frac{d^2}{8}$ and if $a \geq 10$, then $f_a(d) > 0, \forall d$, which contradicts (**) ∎

Lemma 9. *With notations as above, if $\pi < 1 + \frac{d^2}{8}$, then the possibilities are:*

$a = 5$*:* $d = 11, 6$
$a = 7$*:* $d = 13, 10$

$a = 8$*:* $d = 7$
or: $a \le 4$.

Proof: From lemma 8 we may assume $a \le 9$ and the inequality $f_a(d) \le 0$ (see proof of lemma 8); i.e. $d^2(a+1) - 28(a-1)d + 16a^2 - 16a + 16 \le 0$. Solving for the values of a under consideration we obtain:
$a = 5, 4 \le d \le 14$;
$a = 6, 5 \le d \le 15$;
$a = 7, 6 \le d \le 15$;
$a = 8, 7 \le d \le 15$;
$a = 9, 9 \le d \le 14$;
On the other hand, using (*) of the proof of lemma 8:

$$(a-1)b = \frac{(a-1)(d^2 - 7d + 10a - 4) + (a-1)an(3a-5)}{2(3a-5)}$$

$$(a-1)b = n\frac{a(a-1)}{2} + \frac{(a-1)(d^2 - 7d + 10a - 4)}{(6a-10)}$$

It follows that $\frac{(a-1)(d^2-7d+10a-4)}{(6a-10)}$ is an integer. Now among the (a, d) listed above, we take only those for which this further condition holds; this gives the statement of the lemma ∎

Theorem 10. *Let $S \subset \mathbb{P}^4$ be a smooth, non degenerated, rational surface isomorphic to $\mathbb{F}_n$ blown-up at r points $y_1, ..., y_r$. Suppose assumptions (*) (see beginning of this section) are satisfied. Then deg$(S) \le 12$.*

Proof: Assume $d > 12$. By Remark 2, $\pi < 1 + \frac{d^2}{8}$. By Lemma 9, $a \le 4$ or $(a, d) = (7, 13)$. In the first case, we know by Proposition 7 that $d \le 12$. Let's consider the case $(a, d) = (7, 13)$. We use relations 1), ...,3) before Lemma 8. From 2): $\pi - 1 = 6b - 7 - 21n$ (+); from 1): $-r = 13 + 49n - 14b$. Inserting in 3): $2b = 7n + 9$. Finally, from (+): $\pi = 21$. We observe that $21 = G(13, 4)$, hence arguing as in Remark 2, we conclude that S is a.C.M.; but this is impossible ([10]) ∎

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